

Reminders

$$\|A\mathbf{x}\| \leq \|A\| \|\mathbf{x}\| \quad \|AB\| \leq \|A\| \|B\|$$

$$\det(A) = 0 \iff A \text{ singular}$$

1 LU Factorization

$$\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} L \end{bmatrix} \begin{bmatrix} U \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} U \end{bmatrix}^{-1} \left(\begin{bmatrix} L \end{bmatrix}^{-1} \mathbf{b} \right)$$

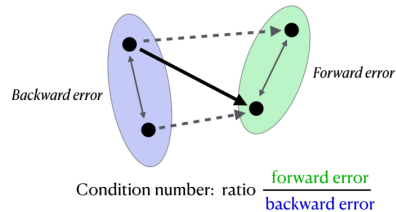
Important: Here, $^{-1}$ does *not* mean: compute a matrix inverse.

Cholesky (SPD)

$A = LL^T$: half the cost/storage of LU. **2×2:**

$$L_{11} = \sqrt{A_{11}}, \quad L_{21} = A_{21}/L_{11}, \quad L_{22} = \sqrt{A_{22} - L_{21}^2}$$

2 Conditioning



Condition number

$$\frac{\|\Delta \mathbf{x}\|}{\|\mathbf{x}\|} \leq \underbrace{\|A^{-1}\| \|A\|}_{=\kappa(A)} \frac{\|\Delta \mathbf{b}\|}{\|\mathbf{b}\|}$$

$$\text{cond}(A) = \kappa(A) = \|A\| \|A^{-1}\|$$

Properties

$$1 \leq \text{cond}(A) \leq \infty$$

$$\text{cond}(A) = \infty \iff A \text{ singular}$$

$$\text{cond}(I) = 1$$

$$\text{cond}(\gamma A) = \text{cond}(A), \quad \gamma \neq 0$$

$$\text{cond}(QA) = \text{cond}(A), \quad Q \text{ rotation}$$

Not the determinant

A tiny determinant does not mean a hard system: a scaled identity has a determinant close to zero, yet it is trivial to solve and perfectly conditioned.

Implications (without proof, good solver, exact b)

$$\frac{\|\Delta \mathbf{x}\|}{\|\mathbf{x}\|} \lesssim \text{cond}(A) \text{ ulp}(1)$$

3 Least Squares

Problem ($A \in \mathbb{R}^{m \times n}$, $m > n$)

$$\min_{\mathbf{x} \in \mathbb{R}^n} \|A\mathbf{x} - \mathbf{b}\|_2^2$$

Normal equations: $A^T A \mathbf{x} = A^T \mathbf{b}$